16.1 Scalars and Vectors

There are two kinds of physical quantities:

(i) those which have only magnitude;
(ii) those -which have both magnitude and direction.

Physical quantities which have only magnitude are called ***scalars***, while physical quantities which have both magnitude and direction are called ***vectors***.

Examples of scalars are length, area, volume, mass, work, energy, distance, speed and density. Examples of vectors are displacement, velocity, acceleration, force and momentum.

Vectors may not necessarily have any particular position associated with them. Such vectors are usually called ***free vectors***. Vectors which have particular positions associated with them may be located along a straight line or through a particular point. Consider town B which is $50km$ from town A where B is North 45° East of A.

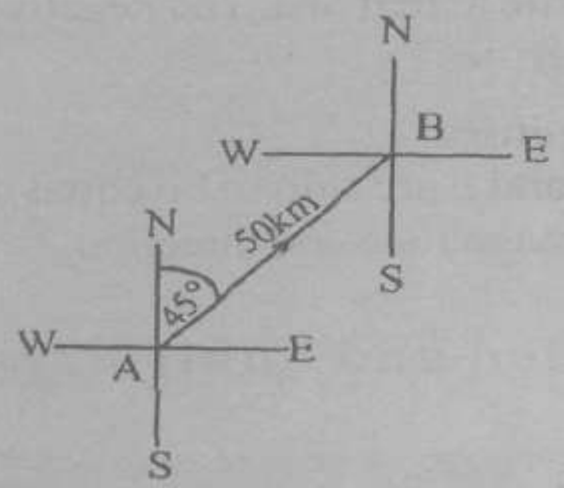


Fig.16.1

The vector 50*km* from *A* in the direction North 45° East can be represented by the line segment $\vec{AB}$. In this respect, a vector can be regarded as a directed line segment.

In Fig. 16.1, the directed line segment has initial point *A* and a terminal point *B*. This is indicated by the arrow on the line segment *AB*. The vector from *A* to *B* as a directed line segment is written $\vec{AB}$.

Notation

In printed work, a vector is shown in heavy print e.g. **a**. In written work a vector is shown by a letter with a bar underneath it, e.g. $\underline{a}$

Magnitude of a Vector

The magnitude of a vector **a**, sometimes called the ***Modulus*** of the vector is represented by |a|.

The Zero Vector

The Zero vector is a vector with a magnitude *of zero.* The zero vector has no particular direction. It is represented by 0 The zero vector is sometimes called a ***null vector***.

The Unit Vector

The unit vector in the direction of the vector *a*. is the vector represented by $\hat{a}$ and is such that

$a = |a|\,\hat{a}$

The Negative Vector

The negative vector of a vector *a* is the vector which has the same magnitude as that of *a* but *a* direction opposite to that of a. The negative vector of *a* is written $-a$.

Equality of Vectors

Two vectors *a* and *b* are said to be equal if they have the same magnitude and direction.

16.2 Addition and Subtraction of Vectors

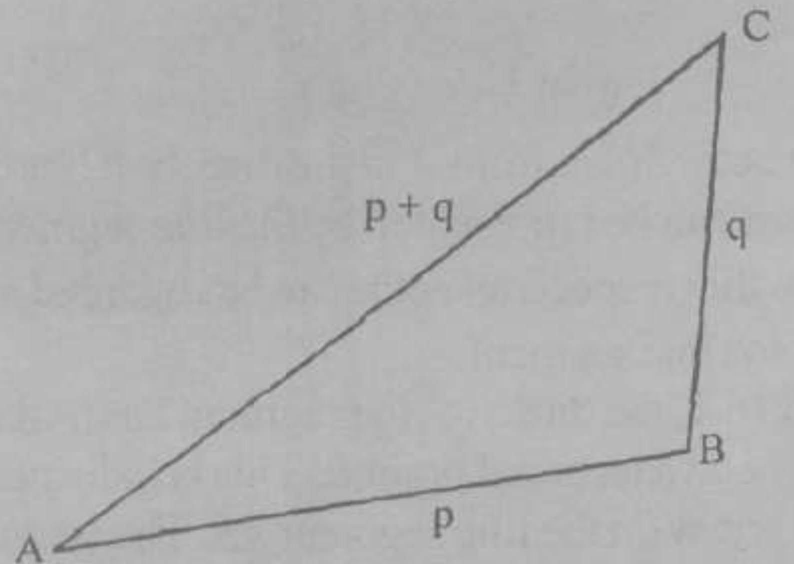


Fig.16.2

The journey from A to C can be accomplished in two ways:

(i) from A to B and then from B to C

(ii) directly from A to C

The vector $\vec{AC}$ is said to be the sum of the vectors $\vec{AB}$ and $\vec{BC}$. We write this as:

$$\vec{AC} = \vec{AB} + \vec{BC}$$

If we represent the vector $\vec{AB}$ by p and the vector $\vec{BC}$ by q then the vector $\vec{AC}$ is represented by the vector $p + q$. Thus:

$$\vec{AC} = p + q$$

The law that $\vec{AC} = \vec{AB} + \vec{BC}$ is called the ***Triangle law of vector addition***. The vector $p + q$ is called the ***resultant*** of the vectors p and q. From the definition of a negative vector we can say that subtraction of a vector is the same as the addition of its negative vector.

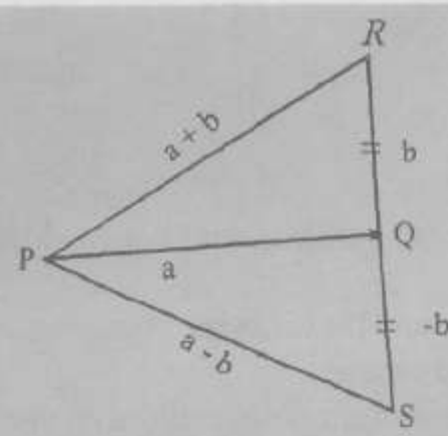


Fig.16.3

From Fig.16.3 $\vec{QS} = -\vec{QR}$

$$\vec{PQ} - \vec{QR} = \vec{PQ} + (-\vec{QR})$$
$$= \vec{PQ} + \vec{RQ}$$
$$= \vec{PQ} + \vec{QS}$$
$$= \vec{PS}$$

Fig.16.3 demonstrates the sum and difference of two vectors.

If $\vec{PQ} = a$, $\vec{QR} = b$, $\vec{QS} = -b$

Then

$$\vec{PR} = a + b$$
$$\vec{PS} = a - b$$

Parallelogram law

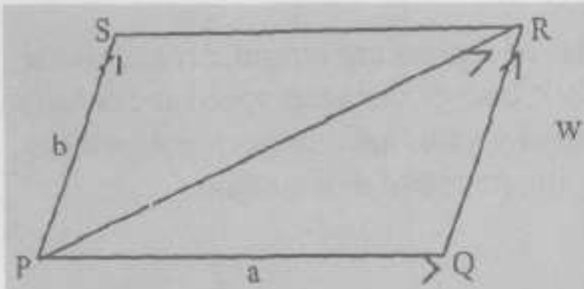


Fig.16.4

The parallelogram law of vector addition states that the resultant of two vectors is represented by the diagonal of the parallelogram whose adjacent sides are the two vectors.
With reference to Fig.16.4

$$\vec{PQ} + \vec{PS} = \vec{PQ} + \vec{QR}$$
$$= \vec{PR}$$

The parallelogram law of vector addition is seen as being equivalent to the Triangle law of vector addition.

Addition of Several Vectors

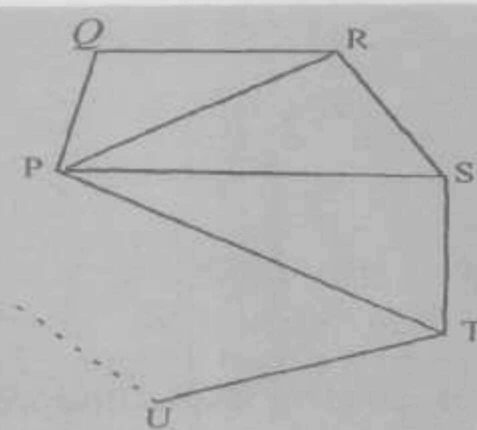


Fig.16.5

Suppose there are more than two vectors to be added. The triangle law of vector addition can be used repeatedly to add more than two vectors. From Fig.16.5,

$$\vec{PQ} + \vec{QR} = \vec{PR}$$
$$\vec{PR} + \vec{RS} = \vec{PS}$$
$$\vec{PS} + \vec{ST} = \vec{PT}$$

Hence,

$\vec{PQ} + \vec{QR} + \vec{RS} + \vec{ST} = \vec{PT}$
This is called the ***Polygon of Vectors***.

Example 1

If ABC is a triangle, what is the sum of the vectors represented by $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$?

Solution

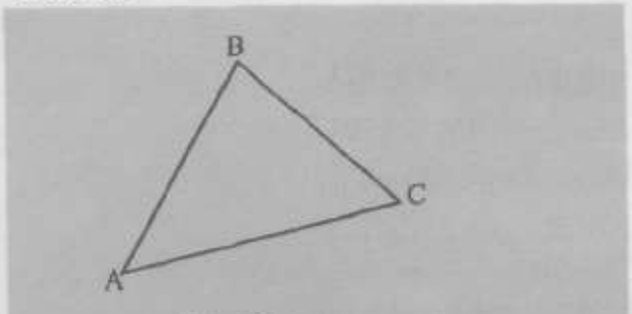


Fig.16.6

$$\begin{aligned}\vec{BC} + \vec{CA} + \vec{AB} &= \vec{BA} + \vec{AB}\\ &= -\vec{AB} + \vec{AB}\\ &= 0\end{aligned}$$

Example 2

If the vectors a, b are represented by the sides $\vec{AB}$, $\vec{AC}$ of a triangle ABC, what vectors are represented by:

(i) $\vec{BC}$ (ii) $\vec{CB}$ (iii) $\vec{AD}$

where D is the mid-point of $\vec{BC}$

Solution

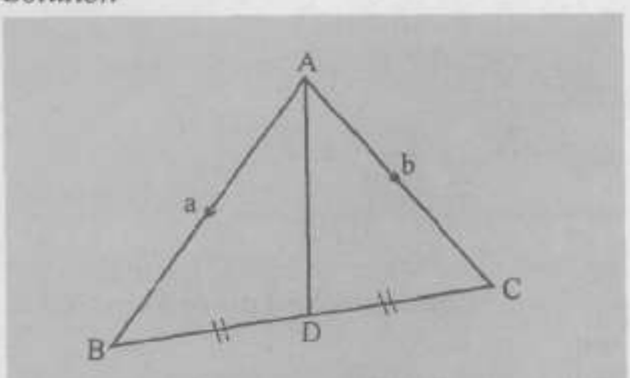


Fig.16.7

(i) $$\begin{aligned}\vec{BC} &= \vec{BA} + \vec{AC}\\ &= -\vec{AB} + \vec{AC}\\ &= -a + b\end{aligned}$$

$$\vec{BC} = b - a$$

(ii) $\vec{CB} = -\vec{BC}$
$= -(\mathbf{b} - \mathbf{a})$
$= -\mathbf{b} + \mathbf{a}$
$= \mathbf{a} - \mathbf{b}$

(iii) $\vec{AD} = \vec{AB} + \vec{BD}$
$= \vec{AB} + \frac{1}{2}\vec{BC}$
$= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$
$= \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$
$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
$= \frac{1}{2}(\mathbf{a} + \mathbf{b})$

Example 3

Show that if a and b are two vectors, then, $a + b = b + a$

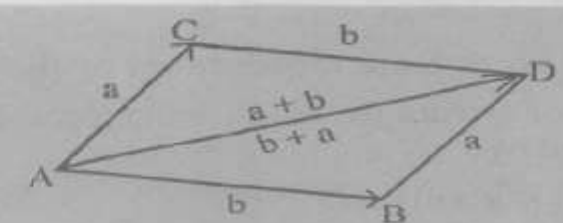


Fig.16.8

$\vec{AD} = \vec{AB} + \vec{BD}$
$= b + a$

Also,

$\vec{AD} = \vec{AC} + \vec{CD}$
$= \mathbf{a} + \mathbf{b}$
$\therefore \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

T h ***Commutative Law of Vector addition.***

We have established the fact that addiction of two vectors is commutative.

Example 4

If a, b and c are three vectors, show that $(a + b) + c = a + (b + c)$.

Solution

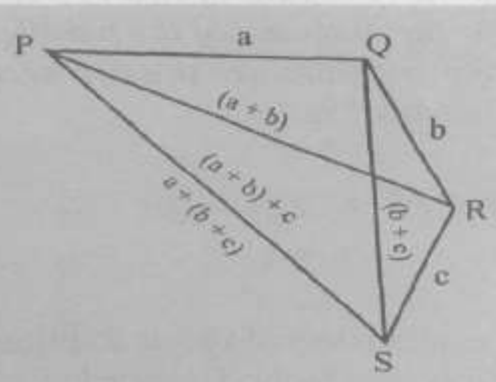


Fig.16.9

Let $\vec{PQ} = a$, $\vec{QR} = b$, and $\vec{RS} = c$
then,

$$\begin{aligned}\vec{PS} &= \vec{PR} + \vec{RS} \\ &= (a + b) + c\end{aligned}$$

Also,

$$\begin{aligned}\vec{PS} &= \vec{PQ} + \vec{QS} \\ &= a + (b + c)\end{aligned}$$

Hence $(a + b) + c = a + (b + c)$
This law is called the ***Associative law of vector addition***.

16.3 Multiplication of a Vector by a Scalar

Let k be a scalar, then ka is defined as a Vector parallel to a and whose magnitude is equal to $k|a|$.

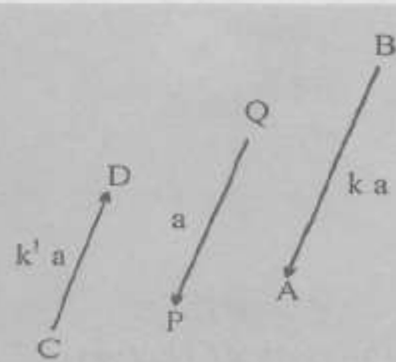


Fig.16.10

In Fig. 16.10

$$\vec{PQ} = a$$
$$\vec{AB} = ka$$
$$\vec{CD} = k!a (0 < k! < 1)$$

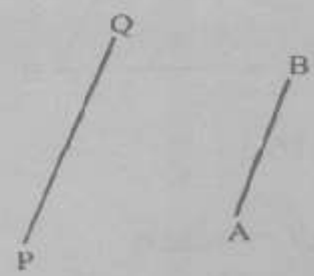


Fig.16.11

In Fig.16.11 if $|\vec{PQ}| = 2|\vec{AB}|$, and $\vec{PQ} \parallel \vec{AB}$ then,

$\vec{PQ} = 2.\vec{AB}$

Example 5

P, Q, R and S are the vertices of a quadrilateral such that A and B are the mid-points of the side $\vec{QR}$ and $\vec{PS}$ respectively. Show that:

$\vec{AB} = \frac{1}{2}(\vec{QP} + \vec{RS})$

Solution

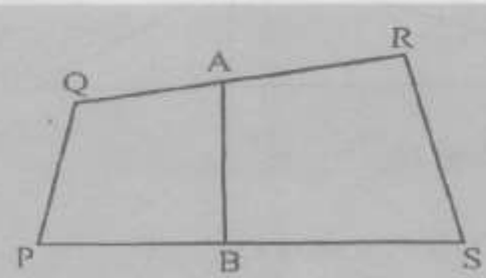


Fig.16.12

$$\vec{AB} = \vec{AQ} + \vec{QP} + \vec{PB}$$

Also, $\vec{AB} = \vec{AR} + \vec{RS} + \vec{SB}$

$$2\vec{AB} = \vec{AQ} + \vec{QP} + \vec{PB} + \vec{AR} + \vec{RS} + \vec{SB}$$
$$= \vec{AQ} + \vec{AR} + \vec{QP} + \vec{RS} + \vec{PB} + \vec{SB}$$

But

$$\vec{AQ} + \vec{AR} = \vec{AQ} - \vec{AQ} = 0$$

Also

$$\vec{PB} + \vec{SB} = \vec{PB} - \vec{PB} = 0$$

Hence,

$$2\vec{AB} = \vec{QP} + \vec{RS}$$

$$\vec{AB} = \frac{1}{2}(\vec{QP} + \vec{RS})$$

Example 6

Prove that the diagonals of a parallelogram bisect each other.

Solution

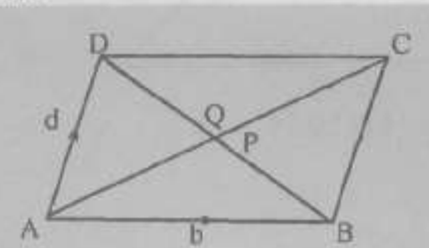


Fig.16.13

In the parallelogram $ABCD$,

Let $\vec{AB} = \text{b}$ and $\vec{AD} = \text{d}$

$\therefore \vec{DC} = \text{b}$ and $\vec{BC} = \text{d}$

$$\vec{AC} = \vec{AB} + \vec{BC} = \text{b} + \text{d}$$

If P is the mid-point of AC, then

$$\vec{AP} = \frac{1}{2}\vec{AC} = \frac{1}{2}(\text{b} + \text{d})$$

$$\vec{BD} = \vec{BA} + \vec{AD}$$
$$= -\text{b} + \text{d}$$

If Q is the mid-point of BD, then

$$\vec{BQ} = \frac{1}{2}\vec{BD} = \frac{1}{2}(-\text{b} + \text{d})$$

$$\vec{AQ} = \vec{AB} + \vec{BQ} = \text{b} + \frac{1}{2}(-\text{b} + \text{d})$$
$$= \frac{1}{2}(\text{b} + \text{d})$$

Thus:

$$\vec{AP} = \vec{AQ}$$
$$\therefore \text{P} \equiv \text{Q}$$

Hence, the diagonals $\vec{AC}$ and $\vec{BD}$ bisect each other.

Example 7

If ABC is a triangle and M and N are the mid-points of AB and BC respectively, show that $MN = \frac{1}{2}AC$

Solution

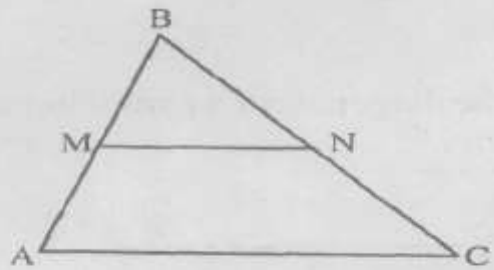


Fig.16.14

$$\vec{MN} = \vec{MB} + \vec{BN}$$

and $\vec{MN} = \vec{MA} + \vec{AC} + \vec{CN}$

$$\therefore 2\vec{MN} = \vec{MB} + \vec{BN} + \vec{MA} + \vec{AC} + \vec{CN}$$

$$= (\vec{MB} + \vec{MA}) + \vec{AC} + (\vec{BN} + \vec{CN})$$

But $\vec{MB} + \vec{MA} = O$ and $\vec{BN} + \vec{CN} = O$

Hence,

$$2\vec{MN} = \vec{AC}$$

$$\therefore \vec{MN} = \frac{1}{2}\vec{AC}$$

16.4 Position Vectors

The position of a point A relative to a point of reference can be specified by a Vector. If the reference point is the origin, the vector $\vec{OA}$ denotes the position of A relative to 0 and it is called the position vector of A relative to O. If we represent $\vec{OA}$ by $\boldsymbol{a}$ then the vector $\boldsymbol{a}$ is the position vector of A.

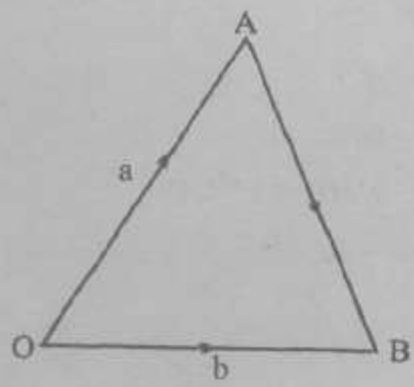


Fig.16.15

Let $\vec{OA} = a$, $\vec{OB} = b$ then

$$\vec{AB} = \vec{AO} + \vec{OB}$$
$$= -\vec{OA} + \vec{OB}$$
$$= -a + b$$
$$= b - a$$

16.5 Components of a Vector in Two Dimensions

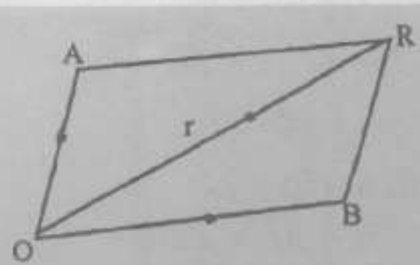


Fig.16.18

We can express the vector $\boldsymbol{r}$ in terms of other vectors which are in the same plane as $\boldsymbol{r}$. Vectors which are in the same plane are called ***Coplanar vectors***.

In Fig. 16.18, let $\vec{OA} = la$, $\vec{OB} = mb$
OBRA is a parallelogram, hence

$$\vec{BR} = \vec{OA} = la$$

$$\vec{OR} = \vec{OB} + \vec{BR}$$

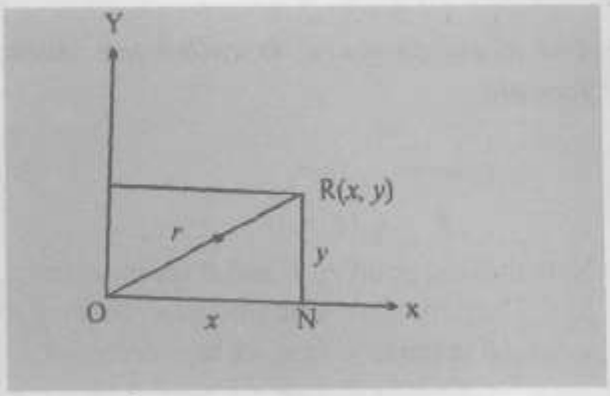
Hence,

$$r = mb + la$$

$$= la + mb$$

The vectors *la* and *mb* are called the components of the-vector $\boldsymbol{r}$ in the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Let us now consider the components of a vector in directions which are mutually perpendicular to each other. We shall refer to the rectangular Cartesian Coordinate System where the position of a point in the Plane is completely specified by its *x* and y coordinates.



Let $\hat{i}$ and $\hat{j}$ be unit vectors in the direction of OX and OY respectively.

Then, $\overrightarrow{ON} = x\hat{i} \quad \overrightarrow{NR} = y\hat{j}$

Let $\overrightarrow{OR} = r$. From Fig. 16.19

$$\overrightarrow{OR} = \overrightarrow{ON} + \overrightarrow{NR}$$

$$= x\hat{i} + y\hat{j}$$

$$\therefore r = x\hat{i} + y\hat{j}$$

Modulus of a vector in terms of its components.

From ΔONR

$$|\overrightarrow{OR}|^2 = x^2 + y^2$$

$$|\overrightarrow{OR}| = \sqrt{x^2 + y^2}$$

$$|r| = \sqrt{x^2 + y^2}$$

$|r|$ gives the ***modulus*** of the vector r.

The vector $r = x\hat{i} + y\hat{j}$ can also be written as $\binom{x}{y}$. The vector $\binom{x}{y}$ is called a ***Column Vector***. Addition, subtraction and scalar multiplication of vectors are done component-wisely.

$$If\ r_1 = x_1\hat{i} + y_1\hat{j}$$

$$r_2 = x_2\hat{i} + y_2\hat{j}$$

and λ a Scalar

then,

$$r_1 + r_2 = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j}$$

$$r_1 - r_2 = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j}$$

$$\lambda r_1 = \lambda x_1\hat{i} + \lambda y_1 j$$

Example 9

Find the modulus of each of the following vectors:

(i) $3\hat{i} + 4\hat{j}$ (ii) $\hat{i} + 3j$

(iii) $-2\hat{i} - 5\hat{j}$ (iv) $\hat{i}\,sin\theta - \hat{j}\,cos\theta$

Solution

(*i*) Let $r_1 = 3\hat{i} + 4\hat{j}$

then,

$$|r_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

(*ii*) *Let* $r_2 = \hat{i} + 3\hat{j}$

$$|r_2| = \sqrt{1 + 9} = \sqrt{10}$$

(*iii*) *Let* $r_3 = -2\hat{i} - 5\hat{j}$

then $r_3 = \sqrt{(-2)^2 + (-5)^2}$

$= \sqrt{4 + 25}$

$= \sqrt{29}$

(*iv*) *Let* $r_4 = \hat{i}\,sin\theta - \hat{j}\,cos\theta$

$|r_4| = \sqrt{sin\theta^2 + (-cos\theta)^2}$

$= \sqrt{(sin^2\theta + cos^2\theta)}$

$= \sqrt{1}$

$= 1$

Example 11

Find the unit vectors in the directions of the following vectors:

(i) $\boldsymbol{r}_1 = 2\hat{\mathrm{i}} + 3\hat{\mathrm{j}}$

(ii) $\boldsymbol{r}_2 = 4\hat{\mathrm{i}} - 5\hat{\mathrm{j}}$

(iii) $\boldsymbol{r}_3 = -3\hat{\mathrm{i}} + \hat{\mathrm{j}}$

(iv) $\boldsymbol{r}_4 = -2\hat{\mathrm{i}} - 2\hat{\mathrm{j}}$

Solution

(i) Let $\hat{r}_1$ be the unit vsector in the direction of $\boldsymbol{r}_1$ then,

$$\hat{r}_1 = \frac{r_1}{|r_1|}$$

$$|r_1| = \sqrt{2^2 + 3^2}$$

$$= \sqrt{13}$$

$$\hat{r}_1 = \frac{1}{\sqrt{13}}(2\hat{\mathrm{i}} + 3\hat{\mathrm{j}})$$

(ii) Let $\hat{r}_2$ be the unit vector in the direction of $\boldsymbol{r}_2$

$$\hat{r}_2 = \frac{r_2}{|r_2|}$$

$$|r_2| = \sqrt{4^2 + (-5)^2}$$

$$= \sqrt{16 + 25}$$

$$= \sqrt{41}$$

$$\hat{r}_2 = \frac{1}{\sqrt{41}}(4\hat{\mathrm{i}} - 5\hat{\mathrm{j}})$$

(iii) Let $\hat{r}_3$ be the unit vector in the direction of $\boldsymbol{r}_3$

$$\hat{r}_3 = \frac{r_3}{|r_3|}$$

$$|r_3| = \sqrt{(-3)^2 + 1^2}$$

$$= \sqrt{9 + 1}$$

$$= \sqrt{10}$$

$\therefore$

$$\hat{r}_3 = \frac{1}{\sqrt{10}}(-3\hat{\mathrm{i}} + \hat{\mathrm{j}})$$

(iv) Let $\hat{r}_4$ be the unit vector in the direction of r_4

$$|r_4| = \sqrt{(-2)^2 + (-2)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$
$$\hat{r}_4 = \frac{1}{\sqrt{8}}(-2\hat{i} - 2\hat{j})$$
$$= \frac{-1}{\sqrt{2}}(\hat{i} + \hat{j})$$